Calculation of $\overline{\alpha}_{\text{QED}}$ on the Z

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We perform a detailed calculation of the hadronic contributions to the running electromagnetic coupling $\overline{\alpha}$ defined on the Z particle (91 GeV). We find for the hadronic contribution, including radiative corrections, $10^5\times\Delta_{\text{had}}\alpha(M_Z^2)=2740\pm12$, or, excluding the top quark contribution, $10^5\times\Delta_{\text{had}}\alpha^{(5)}(M_Z^2)=2747\pm12$. Adding the pure QED corrections, we get a value for the running electromagnetic coupling of $\overline{\alpha}_{\text{QED}}(M_Z^2)=1/(128.965\pm0.017)$.

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I. INTRODUCTION

In a recent paper [1] [de Trocóniz and Ynduráin (TY-I)], we have evaluated the hadronic contributions to the anomalous magnetic moment of the muon; specifically, a very precise determination of the piece involving the photon vacuum polarization function was given there. With a simple change of integration kernel (see below), this analysis can be extended to evaluate the hadronic contribution to the QED running coupling, $\overline{\alpha}_{\text{QED}}(t)$, in particular for $t=M_Z^2$, an important quantity that enters into precision evaluations of electroweak observables. We will find that we can produce a substantial improvement over previous determinations due to our use of complete and correct analyticity and unitarity [2] properties (at low energy), and the high quality of recent Novosibirsk, LEP, and Beijing data.

The running coupling constant may be written as

$$\overline{\alpha}_{\text{QED}}(t)=\frac{e^2/4\pi}{1+\hat{\Pi}(t)},\quad \hat{\Pi}(t)\equiv e^2\Pi_{\text{ren}}(t),\tag{1.1}$$

where e is the electron charge and $\Pi_{\text{ren}}(t)$ is the one-particle-irreducible (1PI) vacuum polarization function, renormalized at $t=0$. To lowest order we can write the shift in α as

$$\overline{\alpha}_{\text{QED}}(t)=\{1+\Delta\alpha(t)\}\frac{e^2}{4\pi},\quad \Delta\alpha(t)=-e^2\Pi_{\text{ren}}(t).$$

In fact, we will evaluate $\Pi_{\text{ren}}(t)$ including the first radiative corrections, so the full Eq. (1.1) has to be used to find the effective coupling. However, we will follow current usage and will write, for the hadronic contributions $\hat{\Pi}_h\equiv e^2\Pi_{\text{ren}}^{\text{had}}$,

$$\Delta_{\text{had}}\alpha\equiv-\hat{\Pi}_h=-e^2\Pi_{\text{ren}}^{\text{had}},$$

or, distinguishing between lowest order (index 0) and next order (index 1),

$$\Delta_{\text{had}}^{(0)}\alpha\equiv-\hat{\Pi}_h^{(0)}=-e^2\Pi_{\text{ren}}^{\text{had}(0)},$$

$$\Delta_{\text{had}}^{(1)}\alpha\equiv-\hat{\Pi}_h^{(1)}=-e^2\Pi_{\text{ren}}^{\text{had}(1)}.$$

By using a dispersion relation, one can write this hadronic contribution at energy squared t, $\hat{\Pi}_h(t)$, as

$$-\hat{\Pi}_h(t)\equiv-e^2\Pi_{\text{ren}}^{\text{had}}(t)=-\frac{t\alpha}{3\pi}\int_{4m(2/\pi)}^{\infty}ds\,\frac{R(s)}{s(s-t)},\tag{1.2a}$$

with

$$R(s)=\frac{\sigma(e^+e^-\to\text{hadrons};s)}{\sigma^{(0)}(e^+e^-\to\mu^+\mu^-;s)},$$

$$\sigma^{(0)}(e^+e^-\to\mu^+\mu^-;s)\equiv\frac{4\pi\alpha^2}{3s},\tag{1.2b}$$

and the integral in Eq. (1.2a) has to be understood as a principal part integral. This is similar to the Brodsky–de Rafael expression for the hadronic vacuum polarization (h.v.p.) contribution to the muon magnetic moment anomaly,

$$a(\text{h.v.p.})=\int_{4m(2/\pi)}^{\infty}ds\,K(s)R(s),$$

$$K(s)=\frac{\alpha^2}{3\pi^2 s}\hat{K}(s),$$

$$\hat{K}(s)=\int_0^1 dx\,\frac{x^2(1-x)}{x^2+(1-x)s/m_\mu^2}.$$

Therefore, we can carry over all the work from TY-I with the simple replacement

$$K(s)\to-\frac{t\alpha}{3\pi}\frac{1}{s(s-t)}.$$

For the coupling at the Z we will take $t=M_Z^2$. Because of this similarity with the $g-2$ calculation, we will dispense with many discussions or details; they may be found in TY-I. Indeed, the present paper should be considered as a sequel to the former one.

After the corresponding evaluations, we find, to next to leading order in α,

$$\begin{aligned}10^5\times\Delta_{\text{had}}\alpha(M_Z^2)&=-10^5\times[\hat{\Pi}_h^{(0)}(M_Z^2)+\hat{\Pi}_h^{(1)}(M_Z^2)]\\&=2740\pm12,\end{aligned}\tag{1.3}$$

or, excluding the top quark contribution,

$$10^5\times\Delta_{\text{had}}\alpha^{(5)}(M_Z^2)=2747\pm12\ \text{(without t)}.\tag{1.4}$$

Adding the known pure QED corrections, the running QED coupling in the momentum scheme is

$$\overline{\alpha}_{\rm QED}(M_Z^2)=\frac{1}{128.965\pm 0.017}. \tag{1.5}$$

II. CONTRIBUTIONS TO THE LOWEST ORDER $-\hat{\Pi}_h^{(0)}$ IN THE ENERGY RANGE FROM THRESHOLD TO 2 GeV^2

A. The region $s\leqslant 1.2$ GeV^2

To zero order in the electromagnetic interactions, we can write $-\hat{\Pi}_h^{(0)}$ as a sum of contributions of different intermediate states in various energy slices. We start with the 2π states for $s\leqslant 1.2\ \mathrm{GeV}^2$. We will subdivide this in turn into two pieces: from threshold, $4m_\pi^2$, to 0.8 GeV^2, and the higher-energy piece.

1. The region below 0.8 GeV^2

We can express $R^{(0)}$ in terms of the pion form factor, F_π:

$$R^{(0)}(s)=\frac{1}{4}\left(1-\frac{4m_\pi^2}{s}\right)^{3/2}|F_\pi(s)|^2, \tag{2.1}$$

where by m_π we understand the charged pion mass. We can also relate F_π to the decay $\tau^+\to\overline{\nu}_\tau\pi^+\pi^0$. Consider the correlator

$$\Pi_{\mu\nu}^V=i\int d^4x\, e^{ip\cdot x}\langle 0|TV_\mu^+(x)V_\nu(0)|0\rangle$$
$$=(-p^2g_{\mu\nu}+p_\mu p_\nu)\Pi^V(s)+p_\mu p_\nu\Pi^S(s),$$
$$s=p^2, \tag{2.2a}$$

with V_μ the weak vector current. Then neglecting isospin breaking (except for the phase-space factor), we have at low s

$$v_1(s)\equiv 2\pi\,\mathrm{Im}\,\Pi^V=\frac{1}{12}\left\{\left[1-\frac{(m_{\pi^+}-m_{\pi^0})^2}{s}\right]\right.$$
$$\left.\times\left[1-\frac{(m_{\pi^+}+m_{\pi^0})^2}{s}\right]\right\}^{3/2}|F_\pi(s)|^2, \tag{2.2b}$$

and, on the other hand, v_1 may be obtained from the experimental measurements of the decay $\tau^+\to\overline{\nu}_\tau\pi^+\pi^0$.

To obtain $F_\pi(s)$, we will fit the recent Novosibirsk data [3] on $e^+e^-\to\pi^+\pi^-$ and the τ decay data of Aleph and Opal [3]. We will take into account, at least partially, isospin breaking effects by allowing different masses and widths for the ρ^0,ρ^+ resonances. Moreover, and to get a good grip in the low-energy region where data are nonexistent or very poor, we also fit $F_\pi(s)$ at spacelike s [3]. This is possible in our approach because we use an expression for F_π that takes fully into account its analyticity properties [2]. To be precise, we use that the phase of $F_\pi(s)$ is equal to that of $\pi\pi$ scattering, in the elastic region, and then the Omnès-Muskhelishvili method. We write

$$F_\pi(s)=G(s)J(s). \tag{2.3a}$$

Here J is expressed in terms of the P-wave $\pi\pi$ phase shift, δ_1^1, as

$$J(s)=e^{1-\delta_1^1(s_0)/\pi}\left(1-\frac{s}{s_0}\right)^{[1-\delta_1^1(s_0)/\pi]s_0/s}\left(1-\frac{s}{s_0}\right)^{-1}$$
$$\times\exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{s_0}ds'\frac{\delta_1^1(s')}{s'(s'-s)}\right\}. \tag{2.3b}$$

s_0 is the energy at which inelasticity starts becoming important (in practice, above the percent level); we will take $s_0=1.1\ \mathrm{GeV}^2$ in actual calculations.

The exponential factor

$$\exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{s_0}ds'\frac{\delta_1^1(s')}{s'(s'-s)}\right\}$$

in Eq. (2.3b) guarantees that the phase of $J(s)$ is equal to $\delta_1^1(s)$ for $s\leqslant s_0$, hence also to the phase of F_π. The rest is included so that J is smooth at $s=s_0$, and has the behavior $|J(s)|\sim 1/s$ at large energies.

Because of this equality of the phase of $J(s)$ and the phase of $F_\pi(s)$ below $s=s_0$, it follows that $G(s)$ will be an analytic function also for $4m_\pi^2\leqslant s\leqslant s_0$, and thus in the whole s plane except in a cut from $s=s_0$ to $+\infty$. If we now make the conformal transformation

$$z=\frac{\frac{1}{2}\sqrt{s_0}-\sqrt{s_0-s}}{\frac{1}{2}\sqrt{s_0}+\sqrt{s_0-s}}, \tag{2.4a}$$

then, as a function of z, G will be analytic in the unit disk and we can thus write a convergent Taylor series for it. Incorporating the condition $G(0)=1$, which follows from $F_\pi(0)=1$, and undoing the transformation, we have

$$G(s)=1+c_1\left[\frac{\frac{1}{2}\sqrt{s_0}-\sqrt{s_0-s}}{\frac{1}{2}\sqrt{s_0}+\sqrt{s_0-s}}+\frac{1}{3}\right]$$
$$+c_2\left[\left(\frac{\frac{1}{2}\sqrt{s_0}-\sqrt{s_0-s}}{\frac{1}{2}\sqrt{s_0}+\sqrt{s_0-s}}\right)^2-\frac{1}{9}\right]+\cdots, \tag{2.4b}$$

where $c_1,c_2,\ldots$ are free parameters. Actually, only two terms will be necessary to fit the data.

Next, to obtain J, and hence F_π, we need a parametrization of $\delta_1^1(s)$. We can use the well-known effective range theory to write

$$\cot\delta_1^1(s)=\frac{s^{1/2}}{2k^3}(m_\rho^2-s)\hat{\psi}(s),\quad k=\frac{\sqrt{s-4m_\pi^2}}{2}, \tag{2.5}$$

where we have extracted the zero corresponding to the ρ resonance. Now the effective range function $\hat{\psi}(s)$ is analytic in the full s plane except for a cut for $[-\infty,0]$ and the inelastic cut $[s_0,+\infty]$. We can profit from this analyticity by making again a conformal transformation into the unit circle, which is now given by

$$w=\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}. \tag{2.6}$$

We can therefore expand $\hat{\psi}$ in a convergent series[1] of powers of w. Undoing the transformation, we then have

$$\delta_1^2(s)=\operatorname{arc\,cot}\left\{\frac{s^{1/2}}{2k^3}(m_\rho^2-s)\left[b_0+b_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}+\cdots\right]\right\}. \tag{2.7}$$

We note that $b_0=\text{const}$, $b_{i\geqslant 1}=0$ would correspond to a pure Breit-Wigner shape for the ρ. By allowing for more terms in the expansion, we are taking into account the known distortions of the Breit-Wigner shape due to the influence of the left and the inelastic cuts of $\hat{\psi}$. For the actual fits, only b_0, b_1, and m_ρ are needed as parameters.

The values of the parameters are obtained by fitting experimental data on $e^+e^-\to\pi^+\pi^-$, data on $\tau^+\to\overline{\nu}_\tau\pi^+\pi^0$ decay, and data on $F_\pi(s)$ at spacelike s (Ref. [3]). We also include in the fit the value of the $\pi\pi$ P-wave scattering length, which we constrain at

$$a_1^1=(38\pm 3)\times 10^{-3}m_\pi^{-3}, \tag{2.8}$$

consistent with $\pi\pi$ scattering results as well as with current algebra calculations. For the free parameters of our fit we find

$$c_1=0.23\pm 0.02,\quad c_2=-0.15\pm 0.03,\quad b_0=1.062\pm 0.005,$$

$$b_1=0.25\pm 0.04,\quad m_{\rho^0}=772.6\pm 0.5\ \text{MeV}. \tag{2.9}$$

We also find, as a byproduct of our fit, the ρ^0 width as well as the mass and the width of the ρ^+, the P-wave scattering length, and the mean-square radius and second coefficient associated with the form factor of the pion:

$$\Gamma_{\rho^0}=147.4\pm 0.8,\quad a_1^1=(41\pm 2)\times 10^{-3}m_\pi^{-3},$$

$$\langle r_\pi^2\rangle=0.435\pm 0.002\ \text{fm},\quad c_\pi=3.60\pm 0.03\ \text{GeV}^{-4}, \tag{2.10a}$$

and

$$m_{\rho^+}=773.8\pm 0.6\ \text{MeV},\quad \Gamma_{\rho^+}=147.3\pm 0.9\ \text{MeV}. \tag{2.10b}$$

The χ^2/d.o.f. (degree of freedom) of the fit is 246/204 with only statistical errors, but improves to 214/204 when experimental systematic errors are included.

We have *not* included in this fit the experimental $\pi\pi$ phase shifts (except for the scattering length), as they are known to suffer from uncertainties associated with the method of extraction: $\pi\pi$ scattering cannot be measured directly. However, we have checked that adding them would not alter substantially our fit or parameters. Details of this, and other aspects of the calculation, may be found in TY-I, where also the results of separate fits to e^+e^- and τ decay data are presented.

With the above parametrization of F_π we can evaluate immediately the corresponding contribution to $\hat{\Pi}$. We find, with self-explanatory notation,

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;2\pi;s\leqslant 0.8\ \text{GeV}^2)=307.6\pm 2.2\pm 2.9. \tag{2.11a}$$

The first error is statistical; the second is a combination of systematic (taking into account the correlations among the various sets of experimental data) and theoretical errors.[2] Equation (2.11a) includes the (small) effect of ω-ρ mixing, evaluated with the standard Gounnaris-Sakurai method as in TY-I.

Errors included in this work are divided into statistical and systematic. Evaluation of the statistical errors is standard: the fit procedure (using the program MINUIT) provides the full error (correlation) matrix at the χ^2 minimum. This matrix is used when calculating the corresponding integral for $\hat{\Pi}$, therefore incorporating automatically all the correlations among the various fit parameters.

In addition, for every energy region, we have considered the errors that stem from experimental systematics, as well as those originating from deficiencies of the theoretical analysis. The experimental systematics covers the errors given by the individual experiments included in the fits. Also, when conflicting sets of data exist, the calculation has been repeated, and the given systematic error bar enlarged to encompass all the possibilities. In general, errors (considered as uncorrelated) have been added in quadrature. The exceptions are explicitly discussed in the text.

2. The $\pi\pi$ contribution in the region $0.8\leqslant s\leqslant 1.2$ GeV^2

For the contribution in the region $0.8\leqslant s\leqslant 1.2\ \text{GeV}^2$, we integrate numerically the experimental data [4] and get

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;2\pi;0.8\leqslant s\leqslant 1.2\ \text{GeV}^2)=27.3\pm 0.3\pm 0.5. \tag{2.11b}$$

With the above result,

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;2\pi;4m_\pi^2\leqslant s\leqslant 1.2\ \text{GeV}^2)=334.9\pm 4.1, \tag{2.12}$$

[1]It is to be noted that this series, as well as that in terms of z above, are quickly convergent in the region of interest for us here, which is mapped in segments contained in $[-0.57,0.24]$ inside the unit circles; see TY-I for details.

[2]If we had used data on e^+e^-, but not on τ decay, we would have obtained a slightly smaller number and a larger error: $-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;2\pi;s\leqslant 0.8\ \text{GeV}^2)=306.5\pm 4.0\pm 4.3$. We will take Eq. (2.11a) to be our best result here.

where both systematic errors (related to the same normalization uncertainty) have been added coherently.

3. The 3π, 2K, and other contributions in the region $s \leqslant 1.2\ \mathrm{GeV}^2$

For the 3π contribution, we fit experimental data [4] with Breit-Wigner formulas (including the correct threshold behavior) for the ω,ϕ resonances, plus a constant. We have two sets of experimental data; the difference between the evaluations with each of them is incorporated into the systematic error. The χ^2/d.o.f. is 63/60. The contribution to $\hat{\Pi}_h^{(0)}$ is

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;3\pi;9m_\pi^2\leqslant s\leqslant 1.2\ \mathrm{GeV}^2)=39.5\pm 0.3\pm 1.5. \tag{2.13}$$

The $2K$ states are treated in the same manner, fitting simultaneously $e^+e^-\to K_LK_S$ and $e^+e^-\to K^+K^-$ data [4] with the same Breit-Wigner parameters for the ϕ; the χ^2/d.o.f. is 84/82. For details, we refer again to TY-I. We get

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;2K;s\leqslant 1.2\ \mathrm{GeV}^2)=41.6\pm 0.2\pm 1.3. \tag{2.14}$$

The contribution of 4π states is evaluated by numerical integration, with the trapezoid rule, of experimental data [5]. The systematic error includes the (estimated) difference between evaluations based on different sets of experimental data. This gives the result

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;4\pi;s\leqslant 1.2\ \mathrm{GeV}^2)=2.6\pm 0.7. \tag{2.15}$$

Finally, $5\pi,6\pi,\eta\pi\pi,\ldots$ states contribute $(0.3\pm 0.2)\times 10^{-5}$ in this region. If we add all the contributions with $s\leqslant 1.2\ \mathrm{GeV}^2$, we find

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;s\leqslant 1.2\ \mathrm{GeV}^2)=418.9\pm 4.6. \tag{2.16}$$

B. The energy range $1.2\leqslant s\leqslant 2\ \mathrm{GeV}^2$

We have now a numerical evaluation obtained from a fit to inclusive $e^+e^-\to$hadrons experimental data [5],

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;1.2\leqslant s\leqslant 2\ \mathrm{GeV}^2)=53.1\pm 5.3. \tag{2.17}$$

III. THE LOWEST ORDER $-\hat{\Pi}_h^{(0)}$ IN THE ENERGY RANGE ABOVE 2 GeV^2: THE FULL $\hat{\Pi}_h^{(0)}(M_Z^2)$

A. QCD calculations

For the QCD calculations,[3] we take the following approximation: away from quark thresholds, and for n_f massless quark flavors with charges Q_f, we write

$$R^{(0)}(t)=3\sum_f Q_f^2\left\{1+\frac{\alpha_s}{\pi}+(1.986-0.115n_f)\left(\frac{\alpha_s}{\pi}\right)^2+\left(-6.64-1.20n_f-0.005n_f^2-1.240\frac{\left(\sum_f Q_f\right)^2}{3\left(\sum_f Q_f^2\right)}\right)\left(\frac{\alpha_s}{\pi}\right)^3\right\}. \tag{3.1a}$$

To this one adds mass and nonperturbative corrections. We take into account the quark mass effect for quarks with running mass $\bar{m}_i(s)$ which correct $R^{(0)}$ by the amount, for each quark,

$$-3Q_i^2\bar{m}_i^2(s)\left\{6+28\frac{\alpha_s}{\pi}+(294.8-12.3n_f)\left(\frac{\alpha_s}{\pi}\right)^2\right\}s^{-1}, \tag{3.1b}$$

$$+3Q_i^2\frac{8\bar{m}_i^4(t)}{7}\left\{-\frac{6\pi}{\alpha_s}+\frac{23}{4}+\left(\frac{2063}{24}-10\zeta(3)\right)\frac{\alpha_s}{\pi}\right\}s^{-2}.$$

Finally, for the condensates we add

$$\frac{2\pi}{3s^2}\left(1-\frac{11}{18}\frac{\alpha_s}{\pi}\right)\langle\alpha_sG^2\rangle\sum_f Q_f^2 \tag{3.1c}$$

and

$$\frac{24\pi^2}{s^2}\left(1-\frac{23}{27}\frac{\alpha_s}{\pi}\right)m_i\langle\bar{\psi}_i\psi_i\rangle. \tag{3.1d}$$

We neglect the condensates corresponding to heavy quarks (c,b) and express those for u,d,s in terms of $f_\pi^2m_\pi^2,f_K^2m_K^2$ using the well-known PCAC (partial conservation of axial vector coupling) relations. The condensate contributions are negligible above $s=3\ \mathrm{GeV}^2$.

Equation (3.1b) will be used when $\bar{m}_i^2\ll s$. In practice, this will mean that the contribution of the correction of order $\bar{m}_i^4/s^2$ is less than 10^{-5}. Near the threshold for heavy quarks c,b,t, i.e., when $v_i^2(s)\ll 1$ [with $v_i(s)=(1-4m_i^2/s)^{1/2}$ the velocity of the quark], we use a nonrelativistic (NR) QCD calculation (see Ref. [7] for details) in which the contribution of quark i is

[3]See Ref. [6] for the calculations of the various pieces.

$$R_i^{\rm NR}=3Q_i^2[1+2c_0(s)]\,\frac{3-v_i^2(s)}{2}\,\frac{\pi C_F\tilde{\alpha}_s}{1-e^{-\pi C_F\tilde{\alpha}_s/v_i}}; \tag{3.2a}$$

$$\tilde{\alpha}_s(s)=\left[1+\frac{(93-10n_f)/36+\gamma_E\beta_0/2}{\pi}\,\alpha_s\right]\alpha_s(s), \tag{3.2b}$$

$$c_0(s)\underset{v_i\to 0}{\simeq}\frac{\beta_0\alpha_s}{4\pi}\left\{\ln\frac{s^{1/2}}{m_iC_F\tilde{\alpha}_s}-1-2\gamma_E\right\}.$$

To this we add the leading nonperturbative correction

$$-\frac{2\pi\langle\alpha_sG^2\rangle}{192m_i^4v_i^6}$$

and consider the effective threshold to occur when this overcomes the contribution (3.2a).

In the intermediate region between $v_i^2\ll 1$ and $m_i^2\ll s$, we use the interpolation given by Schwinger [8],

$$R_i^{\rm Schw}=3Q_i^2v_i(s)\,\frac{3-v_i^2(s)}{2}\left\{1+C_F\left[\frac{\pi}{3v_i}+\frac{3+v_i}{4}\times\left(\frac{\pi}{2}-\frac{3}{4\pi}\right)\right]\alpha_s\right\}. \tag{3.3}$$

Note, however, that Schwinger's interpolation *cannot* be used for $v_i\to 0$ as it underestimates R_i by a factor of 2.

In the QCD calculations, the error labeled "Cond." is found by inserting the variation obtained setting quark and gluon condensates to zero, and that labeled Λ by varying the QCD parameter. Likewise, we label m_i to the error obtained varying the mass m_i. If an error is not given, it will mean that it falls below the 10^{-5} level.

For the parameter Λ, we take the recent determinations [9] that correspond to the value

$$\alpha_s(M_Z^2)=0.117\pm 0.003;$$

to be precise, we have taken (in MeV, and to four loops)

$$\Lambda(s\leqslant m_c^2)=373\pm 80,\quad \Lambda(m_c^2\leqslant s\leqslant m_b^2)=283\pm 50,$$

$$\Lambda(m_b^2\leqslant s\leqslant m_t^2)=199\pm 30,\quad \Lambda(s\geqslant m_t^2)=126\pm 20.$$

For the gluon condensate, we take $\langle\alpha_sG^2\rangle=0.07\ {\rm GeV}^4$. Finally, for the running quark masses we take

$$\overline{m}_s(1\ {\rm GeV})=0.188\ {\rm GeV},\quad \overline{m}_c(\overline{m}_c)=1.44\ {\rm GeV},$$

$$\overline{m}_b(\overline{m}_b)=4.3\ {\rm GeV},\quad \overline{m}_t(\overline{m}_t)=174\ {\rm GeV},$$

and, for the pole masses,

$$m_c=1.867\pm 0.20\ {\rm GeV},\quad m_b=5.022\pm 0.060\ {\rm GeV},$$

$$m_t=174\pm 5\ {\rm GeV}.$$

For the c,b masses, see Refs. [10,11]; for the t quark, Ref. [12].

B. The regions away from quark thresholds

At the lowest-energy region, we find

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;2\leqslant s\leqslant 3\ {\rm GeV}^2)=71.1\pm 0.5(\Lambda)\pm 0.4({\rm Cond.}); \tag{3.4}$$

the justification of the applicability of QCD in this range is the agreement, within errors, of the QCD calculation with the $e^+e^-\to$hadrons data, and with the more precise data coming from $\tau\to\nu_\tau+$hadrons; this may be seen depicted in, e.g., the plots of the Aleph and Opal data in Ref. [3] (more details may be found in TY-I).

Apart from the region $2\leqslant s\leqslant 3\ {\rm GeV}^2$, we can use the perturbative QCD formulas (3.1) and (3.3) for the energy regions $s\geqslant 3\ {\rm GeV}^2$ provided we stay away from heavy quark thresholds. We will thus get, excluding the $J/\psi,\psi'$ resonance contributions (to be discussed below),

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;3\leqslant s\leqslant 3.7^2\ {\rm GeV}^2)=259.1\pm 1.5(\Lambda) \tag{3.5a}$$

(here the contribution of the error induced by the condensates is already negligible). Then,

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;4.6^2\leqslant s\leqslant 10.086^2\ {\rm GeV}^2)=421.3\pm 0.8(\Lambda). \tag{3.5b}$$

We will separate a region around M_Z^2, because we take the principal vale of the integral. We have thus

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;11.2^2\leqslant s\leqslant 20^2\ \ {\rm GeV}^2)=352.2\pm 0.9,$$

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;20^2\leqslant s\leqslant (M_Z-3\ \ {\rm GeV})^2)=1668.9\pm 0.9, \tag{3.5c}$$

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;(M_Z-3\ \ {\rm GeV})^2\leqslant s\leqslant (M_Z+3\ \ {\rm GeV})^2)=29.2\pm 0.5,$$

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;(M_z+3\ \ {\rm GeV})^2\leqslant s\leqslant 348^2\ \ {\rm GeV}^2)=-794.5\pm 0.7.$$

All the errors are due to the variation of the parameter Λ. Finally,

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;360^2\leqslant s\leqslant 400^2\ \ {\rm GeV}^2)=-4.7\pm 0.3, \tag{3.5d}$$

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;400^2\ \ {\rm GeV}^2\leqslant s\to\infty)=-20.8\pm 0.1.$$

In particular, the total top quark contribution above threshold ($360^2\ {\rm GeV}^2\leqslant s\to\infty$) is -6.5.

We note that part of the ranges contain some of the narrow resonances (ψ, Υ, and T families). We will add their contributions individually later on. For the whole perturbative QCD contributions, we have, adding Eq. (3.4) to Eq. (3.5),

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;2\ \ \mathrm{GeV}^2\leqslant s;\mathrm{pQCD})=1982\pm 7. \tag{3.6}$$

Note that we have added the errors *linearly* as they stem from the same variation in the QCD parameter Λ.

C. The threshold regions

We will make two types of calculations. In the first, we take experimental data (when possible, i.e., at the $\bar{c}c$ and $\bar{b}b$ thresholds); in the second, we take the contribution of the resonances lying below threshold from experiment, plus a background given by the contribution of the light quarks (evaluated with perturbative QCD, as above) and use nonrelativistic QCD to evaluate the contribution of the quarks whose threshold we are crossing. Of course, for the $\bar{t}t$ threshold this is all we have.

1. $\bar{c}c$: $J/\psi,\psi'$ and the continuum $3.7^2\leqslant s\leqslant 4.6^2$ GeV^2

We split this into the contribution of the $J/\psi,\psi'$ that we calculate in the narrow width approximation (NWA), and the rest. For the first, we have

$$10^{-5}\times(69.9\pm 4.5)\ \ (J/\psi),$$

$$10^{-5}\times(23.6\pm 2.1)\ \ (\psi').$$

For the remainder we have two possibilities: use a NRQCD calculation (see below) for the heavy quark, which gives

$$10^{-5}\times[73.2\pm 0.3(\Lambda)]\ \ (uds),$$

$$(\mathrm{QCD};3.7^2\leqslant s\leqslant 4.6^2\ \ \mathrm{GeV}^2),$$

$$10^{-5}\times[66\pm 13(m_c)]\ \ (\bar{c}c)\quad(\mathrm{NRQCD}),$$

$$\mathrm{Sum:}\quad 10^{-5}\times(139\pm 13),$$

Total: $10^{-5}\times(233\pm 14)$ (QCD+NRQCD).

Otherwise, we use experimental data above 3.7^2 GeV^2:

$$10^{-5}\times(111.8\pm 0.6\pm 5.5)\ \ (\mathrm{Exp.,BES}):$$

$$3.7^2\leqslant s\leqslant 4.6^2\ \ \mathrm{GeV}^2,$$

Total: $10^{-5}\times(205.3\pm 7.4)$ (Exp., BES).

NRQCD refers to the nonrelativistic QCD calculation with Eq. (3.2); see TY-I and Ref. [7] for details of this type of calculation. BES are the experimental data from Ref. [13]. The first error for them is statistical, the second systematic.

We give a few more details on the calculation with NRQCD, as it is the model for the other threshold regions. The method (QCD+NRQCD) consists in separating the *u, d, s* contribution; the $\bar{c}c$ one is then treated as follows. If a resonance is below the channel for open charm production, which is set at $s=4m_c^2$ (with c the pole mass of the c quark), then it is considered as a bound state, and treated in the NWA. Above $\bar{c}c$ threshold, one uses nonrelativistic QCD. For our choice of c quark mass, both J/ψ and ψ' should be considered to be below threshold.

The reasonable agreement, within errors, between the (QCD+NRQCD) result for the $\bar{c}c$ contributions in the region $3.7^2\leqslant s\leqslant 4.6^2$ GeV^2, $10^{-5}\times(139\pm 13)$, and the result obtained using experimental data only, $10^{-5}\times(112\pm 6)$, gives one confidence to use the same theoretical method of calculation for the other thresholds where the quality of the experimental data is poorer, or these data are lacking. However, for the $\bar{c}c$ region we consider that the results based on experimental data are the best and thus write

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;3.7^2\leqslant s\leqslant 4.6^2\ \ \mathrm{GeV}^2)=205\pm 7. \tag{3.7}$$

2. $\bar{b}b$: Υ,Υ' and the continuum $10.086^2\leqslant s\leqslant 11.2^2$ GeV^2

For the region around the $\bar{b}b$ threshold, we can repeat the calculations as above. First, we add the contribution of the resonances below the $\bar{b}b$ threshold that we calculate in the NWA:

$$10^{-5}\times(5.8\pm 0.2)\ \ (\Upsilon),$$

$$10^{-5}\times(2.1\pm 0.1)\ \ (\Upsilon').$$

For the continuum we find, for a b quark pole mass of [10] $m_b=5.022\pm 0.060$ GeV,

$$10^{-5}\times[57.9\pm 0.1(\Lambda)]\ \ (udsc),$$

$$(\mathrm{QCD};10.086^2\leqslant s\leqslant 11.2^2\ \ \mathrm{GeV}^2),$$

$$10^{-5}\times[8.7\pm 0.6(m_b)]\ \ (\bar{b}b)\ \ \mathrm{NRQCD},$$

$$\mathrm{Sum:}\quad 10^{-5}\times(66.6\pm 0.6).$$

If we had estimated the $\bar{b}b$ contribution saturating with the resonances $\Upsilon'',\ldots,\Upsilon^V$, with electronic widths as given in Ref. [14], we would have gotten 5.2 ± 1.2 instead of the value 8.7 ± 0.6 that we found with the NRQCD calculation. We choose this last as our preferred value and thus write

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;10.086^2\leqslant s\leqslant 11.2^2\ \ \mathrm{GeV}^2)=75\pm 1, \tag{3.8}$$

3. $\bar{t}t$ threshold: T bound states and the continuum $348^2\leqslant s\leqslant 360^2$ GeV^2

The bound states produce a negligible contribution; for the ground state, a second-order QCD calculation[11] gives $\Gamma(T\to e^+e^-)=12.5\pm 1.5$ keV and thus the contribution to $-10^5\times\hat{\Pi}_h^{(0)}$ is of -0.11. For the threshold region, a NRQCD calculation gives, for the t quark contribution, -0.47, while the *udscb* one is -1.41. (Note that in this calculation we are neglecting electroweak interactions, so we treat the t quark as if it were stable.) All together, we find

TABLE I. Summary of contributions to $\Delta_{\rm had}\alpha=-\hat{\Pi}_h$. BW+const denotes a Breit-Wigner fit, including the correct phase-space factors, plus a constant; note that only for the four narrow resonances $J/\psi,\psi',\Upsilon,\Upsilon'$ do we use the NWA. The errors are uncorrelated except those for QCD calculations (that have to be added linearly) and those for the 2π states (see text). For details of the final-state $\gamma+$ hadrons, we refer the reader to TY-I.

Channel	Energy range	Method of calculation	Contribution to $-10^5\times\Delta_{\rm had}\alpha$
$\pi^+\pi^-$	$s\leqslant 0.8$ GeV^2	Fit to $e^+e^-+\tau+$spacelike data	307.6 ± 3.6
$\pi^+\pi^-$	$0.8\leqslant s\leqslant 1.2$ GeV^2	Fit to e^+e^- data	27.3 ± 0.6
3π	$s\leqslant 1.2$ GeV^2	BW+const fit to e^+e^- data	39.5 ± 1.5
$2K$	$s\leqslant 1.2$ GeV^2	BW+const fit to e^+e^- data	41.6 ± 1.3
$4\pi,5\pi,\eta\pi\pi,\ldots$	$s\leqslant 1.2$ GeV^2	Fit to e^+e^- data	2.9 ± 0.7
Inclusive	$1.2\leqslant s\leqslant 2$ GeV^2	Fit to e^+e^- data	53.1 ± 5.3
Inclusive, *uds*	$2\leqslant s\leqslant 3.7^2$ GeV^2	Perturbative QCD	330.2 ± 2.4
$J/\psi,\psi'$		NWA	93.5 ± 5.0
Inclusive	$3.7^2\leqslant s\leqslant 4.6^2$ GeV^2	Fit to e^+e^- data	111.8 ± 5.5
Inclusive, *udsc*	$4.6^2\leqslant s\leqslant 10.086^2$ GeV^2	Perturbative QCD	421.3 ± 0.8
Υ,Υ'		NWA	7.9 ± 0.2
b quark thresh.	$10.086^2\leqslant s\leqslant 11.2^2$ GeV^2	Pert.+nonrelativistic QCD	66.6 ± 0.6
Incl., $udscb(t)$	$11.2^2\leqslant s\leqslant\infty$ (except t thresh.)	Perturbative QCD	1230.3 ± 3.4
t quark thresh.	$348^2\leqslant s\leqslant 360^2$ GeV^2	Pert.+nonrelativistic QCD	-2.0 ± 0.1
$\gamma+$hadrons	Full range	Various methods	8.5 ± 0.9

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;t\ \ \text{thresh.})=-2. \tag{3.9}$$

The error is negligible.

The total contribution of the threshold regions is thus

$$-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2;c,b,t\ \ \text{thresh's})=278\pm7. \tag{3.10}$$

D. The lowest order $\hat{\Pi}_h^{(0)}(M_Z^2)$

Adding all the contributions to $\hat{\Pi}_h^{(0)}(M_Z^2)$, we get

$$10^5\times\Delta_{\rm had}^{(0)}\alpha(M_Z^2)=-10^5\times\hat{\Pi}_h^{(0)}(M_Z^2)=2732\pm12, \tag{3.11}$$

or, excluding the top quark contribution,

$$10^5\times\Delta_{\rm had}^{(0)}\alpha^{(5)}(M_Z^2)=2739\pm12 \quad \text{(without t)}. \tag{3.12}$$

IV. THE RADIATIVE CORRECTIONS, $-\hat{\Pi}_h^{(1)}$; THE FULL $-\hat{\Pi}_h^{(0+1)}$; $\overline{\alpha}_{\rm QED}(M_Z^2)$

A. $-\hat{\Pi}_h^{(1)}$

We have next the contribution of intermediate states containing a photon. At low energy ($s\leqslant 1.2$ GeV^2), we evaluate them individually, and at high energy ($s\geqslant 1.2$ GeV^2) with the parton model. For the second we have a contribution equal to the zero-order one (for which we take the result of the previous section) multiplied by the factor

$$\frac{\sum_f Q_f^4}{\sum_f Q_f^2}\,\frac{3\alpha}{4\pi}.$$

This gives

$$-10^5\times\hat{\Pi}_h^{(1)}(M_Z;s\geqslant 1.2\ \ \text{GeV}^2)=1.4\pm0.1, \tag{4.1}$$

the error depending on what one does in the quark thresholds, especially around the narrow resonances ($J/\pi,\psi',\Upsilon,\Upsilon'$). For the low-energy region, we repeat, with obvious changes, the analysis of TY-I. Only the processes $\pi^+\pi^-\gamma$, $\pi^0\gamma$, and $\eta\gamma$ produce effects at the 10^{-5} level (3.4 ± 0.8, 2.9 ± 0.2, and 0.8 ± 0.1, respectively, in units of 10^{-5}). The first is evaluated in the narrow width approximation, or with a detailed calculation using theoretical formulas that relate $\pi\pi\gamma$ to $\pi\pi$, and taking into account experimental cuts; the details may be found in TY-I. Both methods give essentially the same result. The other two are evaluated in the narrow width approximation, dominated by the $\rho\to\pi^0\gamma$, $\omega\to\pi^0\gamma$, and $\phi\to\eta\gamma$ contributions. In addition, the low energy ($s<0.7^2$ GeV^2)$\pi^0\gamma$ is calculated with a phenomenological coupling $\pi^0\gamma\gamma$, adjusted to reproduce the decay $\pi^0\to\gamma\gamma$. Again, the details are given in TY-I.

Adding all of this to Eq. (4.1), we find

$$-10^5\times\hat{\Pi}_h^{(1)}(M_Z^2)=8.5\pm0.9. \tag{4.2}$$

We summarize our results in Table I.

B. The full $\Delta_{\rm had}\alpha$; the QED coupling on the Z; discussion

Adding then Eq. (4.2) to the lowest-order expression, we get the final result

$$10^5\times\Delta_{\rm had}\alpha(M_Z^2)=-10^5\times[\hat{\Pi}_h^{(0)}(M_Z^2)+\hat{\Pi}_h^{(1)}(M_Z^2)]$$
$$=2740\pm10, \qquad (4.3)$$

or, excluding the top quark contribution,

$$10^5\times\Delta_{\rm had}\alpha^{(5)}(M_Z^2)=2747\pm12. \qquad (4.4)$$

The pure QED corrections amount to (see, e.g., Ref. [15])

$$-10^5\times\hat{\Pi}_{\rm QED}(M_Z^2)=3149.7687. \qquad (4.5)$$

Adding this to Eq. (4.3) and using Eq. (1.1), we get the value for the running electromagnetic coupling,

$$\bar{\alpha}_{\rm QED}(M_Z^2)=\frac{1}{128.965\pm0.017}. \qquad (4.6)$$

When comparing with other determinations, we will restrict ourselves to those performed *after* the data from Novosibirsk [3,4] and Beijing [13] have become available; these experiments increase the set of data by almost one order of magnitude, and have much better precision than the older ones. Thus, one may consider older determinations as superseded. So we compare our results with the determinations of Refs. [15,16]. We have

$$10^5\times\Delta_{\rm had}\alpha^{(5)}(M_Z^2)=\begin{cases}2743\pm19/2765\pm21 & ({\rm MOR}),\\ 2761\pm36 & ({\rm BP}),\\ 2790\pm40 & ({\rm J}),\end{cases} \qquad (4.7)$$

In the MOR determination, the two values depend on the method of calculation used. As a general rule, comparing with the results quoted above, the reason for a reduced error bar is mostly due to a wider use of perturbative QCD. For instance, the analysis of MOR uses perturbative QCD in the regions $2.8\leqslant s\leqslant3.7^2$ GeV^2 and $s\geqslant5^2$ GeV^2. We have used pertubative QCD in the region $2\leqslant s\leqslant3.7^2$ GeV^2, justified in view of its agreement with the precise new experimental data (as discussed in the text), and the regions above the $\bar{c}c$ and $\bar{b}b$ thresholds. Here, the use of the calculations incorporating the exact effect of the quark masses has enabled us to get a precise determination for $4.6^2\leqslant s\leqslant5^2$ GeV^2, where the experimental data are not very precise, but (again) are perfectly consistent with QCD.

In conclusion, we have performed a detailed evaluation of the hadronic contributions to the running electromagnetic coupling, obtaining a substantially reduced error bar. The ingredients are the following: First, we use Novosibirsk (e^+e^-) and LEP (τ) data to fit the 2π contribution. Invoking the analyticity and unitarity properties of the pion form factor allows us to include spacelike data also, improving the compatibility of the e^+e^- data with the results from τ decay, and reducing the corresponding error. Second, the low-energy 3π and $2K$ states have been considered individually, after the latest Novosibirsk data on the ω and ϕ resonances. We perform a full-fledged fit, including the exact threshold factors. Third, we have used perturbative QCD in the region $s\geqslant2$ GeV^2 (away from quark thresholds). In particular, the recent LEP $\tau\to\nu_\tau+$hadrons and BES $e^+e^-\to$hadrons data justify the QCD result for $2\leqslant s\leqslant3^2$ GeV^2, implying the largest part of the error reduction. We also use the Beijing [13] data, thus gaining precision, for the contribution in the energy range 3.7^2–4.6^2 GeV^2. Last but not least, the next-order radiative corrections have been taken into account. This is essential for our calculation as the radiative contribution is indeed of the same order of the final error bar.

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